R8253

Sub. Code
511401

M.Sc. DEGREE EXAMINATION, APRIL – 2023

Fourth Semester

Mathematics

FUNCTIONAL ANALYSIS

(CBCS – 2019 onwards)

Time : 3 Hours   Maximum : 75 Marks

Part A $(10 \times 2 = 20)$

Answer **all** questions.

1. Define Compactness.

2. Show that the norm function is continuous.

3. State Orthogonality.

4. Give an example of a normed space which is not an inner product space.

5. Define sesquilinear form.

6. Show that $0^* = 0$ and $l^* = l$

7. Define Adjoint operator T^*.

8. What are the adjoints of zero and identity operator?

9. Define weak convergence.

10. Show that an open mapping need not map closed sets onto closed sets.

Part B $(5 \times 5 = 25)$

Answer **all** questions, choosing either (a) or (b).

11. (a) Prove that in a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

Or

(b) State and prove Riesz's lemma.

12. (a) Prove that an orthonormal set is linearly independent.

Or

(b) If Y is a closed subspace of a Hilbert space H, then show that $H = Y \oplus Z$.

13. (a) Prove that the product of two bounded self-adjoint linear operators is a linear operator if and only if the operators commute.

Or

(b) Let the operator $U : H \to H$ and $V : H \to H$ be unitary; H is a Hilbert space. Then prove that:

(i) U is isometric; thus $\|Ux\| \equiv \|x\|$ for all $x \in H$;

(ii) $\|U\| \equiv 1$, provided H $\neq \{0\}$,

(iii) $U^{-1} (= U^*)$ is unitary,

(iv) UV is unitary.

(v) U is normal.

14. (a) State and prove every finite dimensional normed space is reflexive.

Or

(b) If the dual space x' of a normed space X is separable, then prove that X itself is separable.

15. (a) Let X = C[0,1] and

$T : D(T) \to X$

$x \mapsto x'$

Where the prime denotes differentiation and *D(T)* is the subspace of function $x \in X$ which have a continuous derivative. The prove that T is not bounded, but is closed.

Or

(b) Let X be a normed space, If dim $X < \infty$, then show that weak convergence of a sequence implies strong convergence.

Part C $(3 \times 10 = 30)$

Answer any **three** questions.

16. Prove that every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.

17. State and prove Bessel's inequality.

18. Prove that the Hilbert-adjoint operator T^* of a bounded linear operator T: $H_1 \to H_2$ exists, is unique and is a bounded linear operator with norm $\|T^*\| = \|T\|$

19. State and prove Hahn-Banach theorem.

20. A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Then show that if T is bijective, T^{-1} is continuous and bounded.

R8254

Sub. Code
511402

M.Sc. DEGREE EXAMINATION, APRIL – 2023

Fourth Semester

Mathematics

PROBABILITY AND STATISTICS

(CBCS – 2019 onwards)

Time : 3 Hours Maximum : 75 Marks

Part A $(10 \times 2 = 20)$

Answer **all** questions.

1. Define conditional probability.

2. From an ordinary deck of playing cards, cards are to be drawn successively, at random and without replacement. What is the probability that the third spade appears on the sixth draw?

3. Define random vector and give an example.

4. Define the expected value of a random vector.

5. Write the formula for binomial pmf.

6. Show that the sum of independent random variables having Poisson distribution is Poisson distribution.

7. What is the mean and variance of the t-distribution?

8. Define random sample.

9. Define converges in probability.

10. Define limiting distribution.

Part B (5 × 5 = 25)

Answer **all** questions, choosing either (a) or (b).

11. (a) Derive the law of total probability.

Or

(b) For any random variable, show that $P[X = x] = F_X(x) - F_X(x-)$ for all $x \in R$, where $F_X(x-) = \lim_{z \uparrow x} F_X(z)$.

12. (a) Let X_1 and X_2 have the pdf $f(x_1, x_2) = \begin{cases} 8x_1 x_2 & 0 < x_1 < x_2 < 1 \\ 0 & otherwise \end{cases}$. Find $E(X_1 X_2^2)$ and $E(7X_1X_2^2 + 5X_2)$.

Or

(b) Let (X_1, X_2) have the joint cdf $F(x_1, x_2)$ and let X_1 and X_2 have the marginal cdfs $F_1(x_1)$ and $F_2(x_2)$, respectively. Then show that X_1 and X_2 are independent if and only if $F(x_1, x_2) = F_1(x_1) F_2(x_2)$ for all $(x_1, x_2) \in R^2$.

13. (a) Let X be a random variable such that $E(X^m) = \frac{(m+3)!}{3!} 3^m$, $m = 1, 2, 3, ...$ Then find the mgf of X.

Or

(b) Suppose X has a $\chi^2(r)$ distribution. If $k > -r/2$, show that $E(X^k)$ exists and $E(X^k) = \frac{2^k \Gamma\left(\frac{r}{2} + k\right)}{\Gamma\left(\frac{r}{2}\right)}$ if $k > -r/2$.

14. (a) Derive the moments of F-distribution.

Or

(b) Let Y_1, Y_2, Y_3 be the order statistics of a random sample of size 3 from a distribution having pdf

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

Calculate the pdf of the sample range $Z_1 = Y_3 - Y_1$.

15. (a) State and prove the weak law of large numbers.

Or

(b) Suppose $X_n \xrightarrow{P} X$ and a is a constant. Then show that $aX_n \xrightarrow{P} aX$.

Part C (3 × 10 = 30)

Answer any **three** questions.

16. Let X be a continuous random variable with pdf $f_x(x)$ and support of X. Let $Y = g(X)$, where g(x) is a one-to-one differentiable function, on the support of X, S_X. Denote the inverse of g by $x = g^{-1}(y)$ and let $dx / dy = d[g^{-1}(y)] / dy$. Then show the pdf of Y is given by

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|, \text{ for } y \in S_Y$$

where the support of Y is the set $S_Y = \{y = g(x) : x \in S(X)\}$.

17. State and prove the condition of the equvilance for the independance of two random variables.

18. Explain in detail about the Dirichlet distribution.

19. Let $Y_1 < Y_2 < ... < Y_n$ denote the n order statistics based on the random sample $X_1, X_2 ..., X_n$ from a continuous distribution with pdf $f(x)$ and support (a, b). Then derive the joint pdf of $Y_1, Y_2, ..., Y_n$.

20. Let T_n have a t-distribution with n degrees of freedom, n = 1, 2, 3, and its cdf is

$$F_n(t) = \int_{-\infty}^{t} \frac{\Gamma[(n+1)/2]}{\sqrt{\pi n}\,\Gamma(n/2)} \frac{1}{(1 + y^2/n)^{(n+1)/2}} dy,$$

where the integrand is the pdf $f_n(y)$ of T_n. Show that T_n has a limiting standard normal distribution.

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R8255

Sub. Code
511403

M.Sc. DEGREE EXAMINATION, APRIL – 2023

Fourth Semester

Mathematics

GRAPH THEORY

(CBCS – 2019 onwards)

Time : 3 Hours Maximum : 75 Marks

Part A ($10 \times 2 = 20$)

Answer **all** questions.

1. What is Subgraph?

2. List out three properties of trees.

3. Define Hamiltonian cycle.

4. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunnelling through all of the 27 $1 \times 1 \times 1$ subcubes. If he starts at one corner and always moves on to an uneaten subcube, can he finish at the centre of the cube?

5. Show that it is impossible, using 1×2 rectangles to exactly cover an 8×8 square from which two opposite 1×1 corner squares have been removed.

6. Define edge independence number.

7. Define k-vertex colouring.

8. Define Chromatic number.

9. Distinguish between Planar and non-planar graphs.

10. Prove all planar embeddings of a given connected planar graph have the same number of faces.

Part B (5 × 5 = 25)

Answer **all** questions, choosing either (a) or (b).

11. (a) Prove that an edge e of G is a cut edge of G if and only if e is contained in no cycle of G.

Or

(b) Prove that a vertex v of a tree G is a cut vertex of G if and only if $d(v) > 1$.

12. (a) If G is a simple graph with $v \leq 3$ and $\delta \leq v/2$, then prove G is Hamiltonian.

Or

(b) Prove that $c(G)$ is well defined.

13. (a) Show that a matching M in G is a maximum matching if and only if G contains no M-augmenting path.

Or

(b) Prove that in a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

14. (a) If G is bipartite, then prove $\chi' = \Delta$.

Or

(b) In a critical graph, Prove no vertex cut is a clique.

15. (a) Let G be a simple plane triangulation with $v \geq 4$. Show that G^* is a simple 2-edge-connected 3-regular planar graph.

Or

(b) Prove K_5 is nonplanar.

Part C (3 × 10 = 30)

Answer any **three** questions.

16. Prove that a graph is bipartite if and only if contains no odd cycle.

17. Prove that a graph G with $v \geq 3$ is 2-connected if and only if any two vertices of G are connected by atleast two internally-disjoint paths.

18. Prove that $r(k,k) \geq 2^{k/2}$.

19. Let G be a k-critical graph with a 2-vertex cut $\{u,v\}$. Then prove

    (a) $G = G_1 \cup G_2$ where G_1 is a $\{u,v\}$- component of type $i(i = 1,2)$, and

    (b) both $G_1 + uv$ and $G_2 . uv$ are k-critical (where $G_2 . uv$ denotes the graph obtained from G_2 by identifying u and v)

20. Prove that the following three statements are equivalent

    (a) every planar graph is 4-vertex-colourable;

    (b) every plane graph is 4-face-colourable

    (c) every simple 2-edge-connected 3-regular planar graph IS 3-edgecolourable.